9-2 Taylor Series

Learning Objectives:

I can construct a Taylor (or MacLaurin) series that models a given function

Given the Taylor Series for a function(s), I can write the Taylor series for a other functions that are compositions or products of those function.

Feb 22-2:56 PM

In Groups, do exploration #1 on page 484

Feb 22-2:56 PM

Ex1. Given the function f(x)=cos(x)

a.) Write a tangent line approximation for f(x)=cos(x) at x = 0.

$f(x) = \cos x$

@ $x = 0$

$f(0) = \cos(0)$

$f(0) = 1$

$f'(x) = -\sin x$

$f'(0) = -\sin(0) = 0$

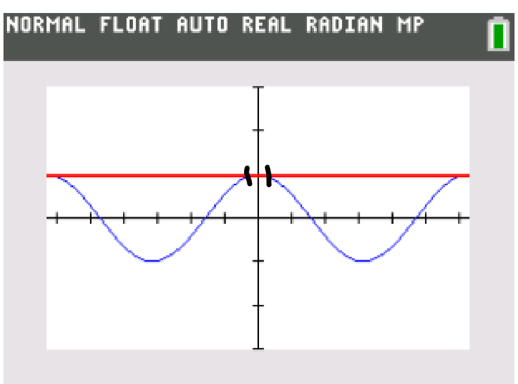
$y - y_1 = m(x - x_1)$

$y - 1 = 0(x - 0)$

$y - 1 = 0$

$y = 1$

$y = 1 + 0x$



b.) A quadratic approximation would be better. Write a quadratic that approximates f(x)=cos(x) at x = 0.

$y = \cos x$

	@ x = 0
$f = \cos x$	1
$f' = -\sin x$	0
$f'' = -\cos x$	-1

$y = a_0 + a_1 x + a_2 x^2$

$a_0 = 1$

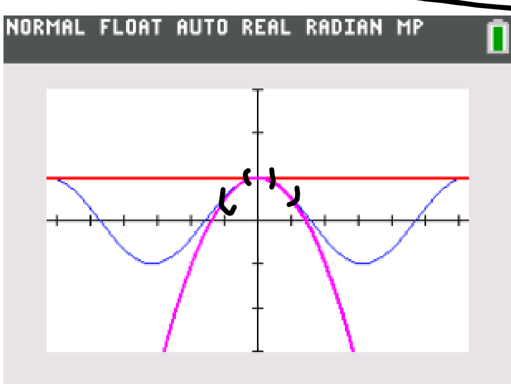
$y' = a_1 + 2a_2 x$

$a_1 = 0$

$y'' = 2a_2$

$a_2 = \frac{-1}{2}$

$$y = 1 + 0x + \frac{-1}{2}x^2$$



Feb 22-2:56 PM

c.) If a quadratic was better than a linear. Then maybe a cubic would better yet and maybe a quartic would be even better. Write a 4th power polynomial that approximates f(x)=cos(x) at x = 0.

	@ x = 0
f = cos x	1
f' = -sin x	0
f'' = -cos x	-1
f''' = sin x	0
f'''' = cos x	1

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$a_0 = 1 \quad a_1 = 0 \quad a_2 = -\frac{1}{2}$

$a_3 = 0$

$4a_4 x^3$

$12a_4 x^2$

$24a_4 x$

$24a_4$

$1 = 24a_4$

$a_4 = \frac{1}{24}$

$y = 1 + 0x + -\frac{1}{2}x^2 + 0x^3 + \frac{1}{24}x^4$

$y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

Taylor Series centered at x = 0

Let f(x) be a function with derivatives of all order throughout some open interval containing 0, then the Taylor Series generated by f(x) at x = 0 is:

$$\frac{f(0)}{0!}+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots+\frac{f^n(0)}{n!}x^n+\ldots$$

The partial sum:

$$P_N(x)=\sum_{k=0}^{n}\frac{f^k(0)}{k!}x^k=\frac{f(0)}{0!}+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots+\frac{f^n(0)}{n!}x^n$$

Is called the nth degree Taylor Polynomial centered at x=0.

Feb 22-2:56 PM

Ex2. Generate the 11th degree Taylor Polynomial for f(x) = ½sin(x) + 1 centered at x=0

$f(x) = \frac{1}{2}\sin x + 1$ $\quad f(0) = 1$

$f'(x) = \frac{1}{2}\cos x$ $\quad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{2}\sin x$ $\quad f''(0) = 0$

$f'''(x) = -\frac{1}{2}\cos x$ $\quad f'''(0) = -\frac{1}{2}$

$f^{IV}(x) = \frac{1}{2}\sin x$ $\quad f''''(0) = 0$

$f^{V}(x) = \frac{1}{2}\cos x$ $\quad f^{V}(0) = \frac{1}{2}$

$f^{VI}(x) = -\frac{1}{2}\sin x$ $\quad f^{VI}(0) = 0$

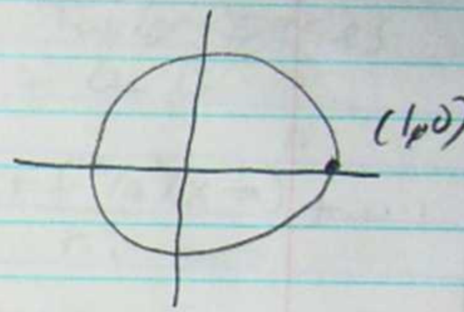
$f^{VII}(x) = -\frac{1}{2}\cos x$ $\quad f^{VII}(0) = -\frac{1}{2}$

$f^{VIII}(x) = \frac{1}{2}\sin x$ $\quad f^{VIII}(0) = 0$

$f^{IX}(x) = \frac{1}{2}\cos x$ $\quad f^{IX}(0) = \frac{1}{2}$

$f^{X}(x) = -\frac{1}{2}\sin x$ $\quad f^{X}(0) = 0$

$f^{XI}(x) = -\frac{1}{2}\cos x$ $\quad f^{XI}(0) = -\frac{1}{2}$



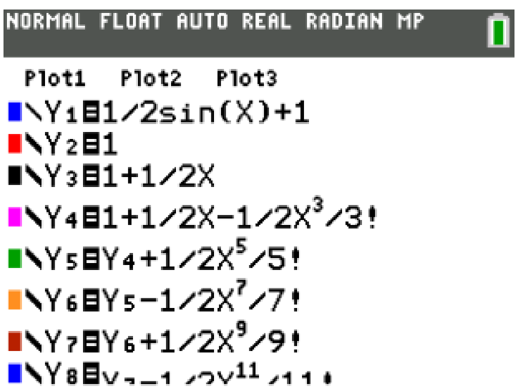
graph f(x) and $P_{11}(x)$

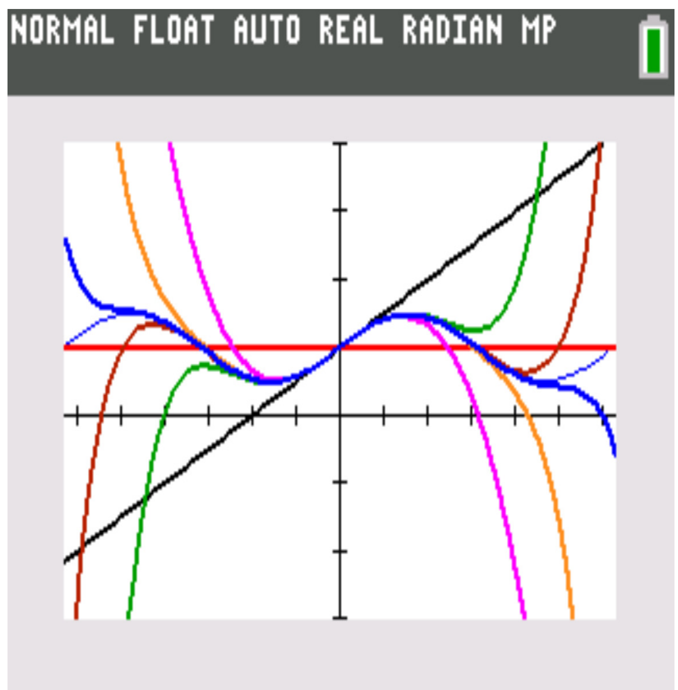
$$P_{11}(x) = 1 + \frac{\frac{1}{2}x}{1!} + \frac{0x^2}{2!} + \frac{-\frac{1}{2}x^3}{3!} + \frac{0x^4}{4!} + \frac{\frac{1}{2}x^5}{5!} + \frac{0x^6}{6!} + \frac{-\frac{1}{2}x^7}{7!} + \frac{0x^8}{8!} + \frac{\frac{1}{2}x^9}{9!} + \frac{0x^{10}}{10!} + \frac{-\frac{1}{2}x^{11}}{11!}$$

$$= 1 + \frac{1}{2}x + -\frac{1}{12}x^3 + \frac{1}{240}x^5 + -\frac{1}{10080}x^7 + \frac{1}{725760}x^9 + -\frac{1}{79833600}x^{11}$$

Graph the function and the Taylor Polynomial

Gut instinct – what do you think is the interval of convergence?





Feb 22-2:56 PM

Taylor Series centered at x = a

Let f(x) be a function with derivatives of all order throughout some open interval containing a, then the Taylor Series generated by f(x) at x = a is:

$$\frac{f(a)}{0!}+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^n(a)}{n!}(x-a)^n+\ldots$$

The partial sum:

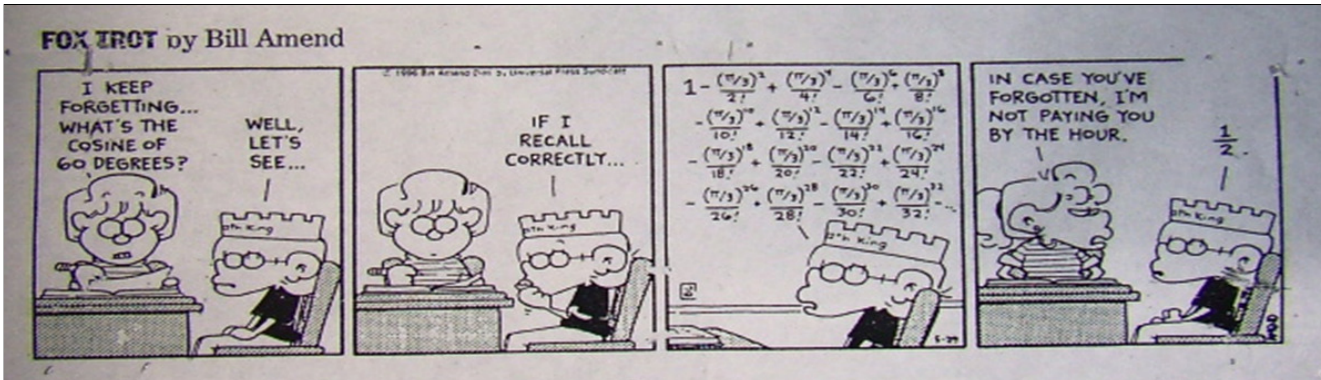
$$P_N(x)=\sum_{k=0}^{n}\frac{f^k(a)}{k!}(x-a)^k=\frac{f(a)}{0!}+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^n(a)}{n!}(x-a)^n$$

Is called the nth degree Taylor Polynomial centered at x=a.

Feb 22-2:56 PM

*Note: A Taylor Series centered at $x = 0$ is also called a McLauren Series. A McLauren Series is just a special case of a Taylor Series.

Feb 22-2:56 PM


FOX TROT by Bill Amend
I KEEP FORGETTING... WHAT'S THE COSINE OF 60 DEGREES?
WELL, LET'S SEE...
IF I RECALL CORRECTLY...
$1-\frac{(\pi/3)^2}{2!}+\frac{(\pi/3)^4}{4!}-\frac{(\pi/3)^6}{6!}+\frac{(\pi/3)^8}{8!}-\frac{(\pi/3)^{10}}{10!}+\frac{(\pi/3)^{12}}{12!}-\frac{(\pi/3)^{14}}{14!}+\frac{(\pi/3)^{16}}{16!}-\frac{(\pi/3)^{18}}{18!}+\frac{(\pi/3)^{20}}{20!}-\frac{(\pi/3)^{22}}{22!}+\frac{(\pi/3)^{24}}{24!}-\frac{(\pi/3)^{26}}{26!}+\frac{(\pi/3)^{28}}{28!}-\frac{(\pi/3)^{30}}{30!}+\frac{(\pi/3)^{32}}{32!}-\cdots$
IN CASE YOU'VE FORGOTTEN, I'M NOT PAYING YOU BY THE HOUR.
$\frac{1}{2}$.

Feb 22-3:02 PM

Ex3. Generate the 6th degree Taylor Polynomial for y=ln(x) centered at x=1. Write the formula for the Taylor Series.

$f(x) = \ln x \qquad f(1) = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = -2$

$f''''(x) = -\frac{6}{x^4} \qquad f''''(1) = 6$

$f^{V}(x) = \frac{24}{x^5} \qquad f^{V}(1) = -24$

graph $f(x)$ and $P_6(x)$

$$P_6(x) = 0 + \frac{1(x-1)}{1!} + \frac{-1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \frac{-6(x-1)^4}{4!} + -\frac{24(x-1)^5}{5!}$$

$$= (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 + \tfrac{1}{4}(x-1)^4 - \tfrac{1}{5}(x-1)^5$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}(x-1)^{n+1}$$

Graph the function and the Taylor Polynomial

Gut instinct – what do you think is the interval of convergence?

NORMAL FLOAT AUTO REAL RADIAN MP

Plot1 Plot2 Plot3

$Y_1 = \ln(X)$

$Y_2 = (X-1)$

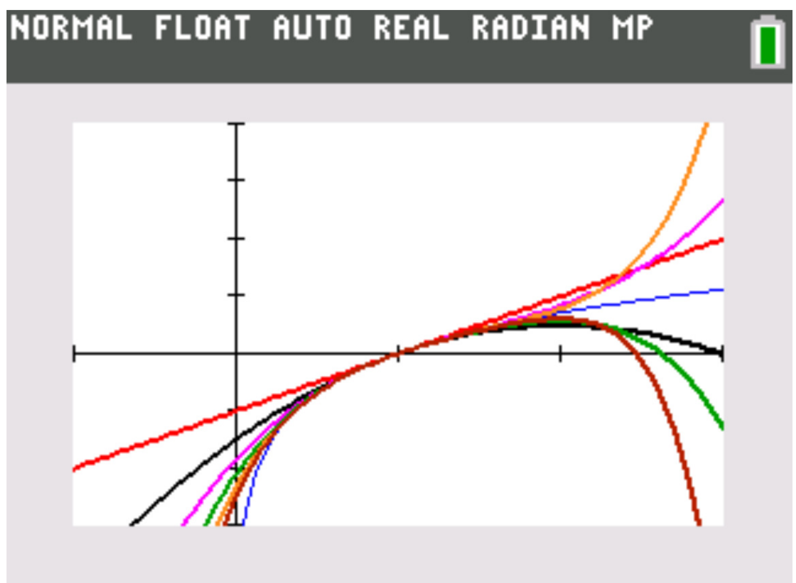
$Y_3 = Y_2 - 1/2(X-1)^2$

$Y_4 = Y_3 + 1/3(X-1)^3$

$Y_5 = Y_4 - 1/4(X-1)^4$

$Y_6 = Y_5 + 1/5(X-1)^5$

$Y_7 = Y_6 - 1/6(X-1)^6$



Ex4. Given the Taylor Series for y=ln(x) centered at x=1 is:

$$\ln(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}(x-1)^{n+1} = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots$$

Find: a.) $g(x) = \ln(x^2)$

b.) $h(x)=x\ln(x)$

a.) $g(x) = \ln(x^2)$

$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots$

$\ln(x^2) = (x^2-1) - \frac{1}{2}(x^2-1)^2 + \frac{1}{3}(x^2-1)^3 - \frac{1}{4}(x^2-1)^4 + \ldots$

b.) $h(x) = x\ln x$

$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots$

$x\ln x = x(x-1) - \frac{1}{2}x(x-1)^2 + \frac{1}{3}x(x-1)^3 - \frac{1}{4}x(x-1)^4 + \ldots$

Homework

Pg 492 # 1-3, 5, 7, 8, 10, 13, 14, 22, 24-26, 31, 36-42

Feb 22-2:56 PM